Centrifugal (centripetal) and Coriolis' velocities and accelerations in spaces with affine connections and metrics as models of space-time

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Abstract

The notions of centrifugal (centripetal) and Coriolis' velocities and accelerations are introduced and considered in spaces with affine connections and metrics [$(\overline{L}_n, g)$-spaces] as velocities and accelerations of flows of mass elements (particles) moving in space-time. It is shown that these types of velocities and accelerations are generated by the relative motions between the mass elements. They are closely related to the kinematic characteristics of the relative velocity and relative acceleration. The centrifugal (centripetal) velocity is found to be in connection with the Hubble law. The centrifugal (centripetal) acceleration could be interpreted as gravitational acceleration as it has been done in the Einstein theory of gravitation. This fact could be used as a basis for working out of new gravitational theories in spaces with affine connections and metrics.

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1 Introduction

1. The relative velocity and the relative acceleration between particles or mass elements of a flow are important characteristics for describing the evolution and the motion of a dynamic system. On the other side, the kinematic characteristics related to the relative velocity and the relative acceleration (such as deformation velocity and acceleration, shear velocity and acceleration, rotation velocity and acceleration, and expansion velocity and acceleration) characterize specific relative motions of particles and /or mass elements in a flow [1]÷[4]. On the basis of the links between the kinematic characteristics related to the relative

velocity and these related to the relative acceleration the evolution of a system of particles or mass elements of a flow could be connected to the geometric properties of the corresponding mathematical model of a space or space-time [5]. Many of the notions of classical mechanics or of classical mechanics of continuous media preserve their physical interpretation in more comprehensive spaces than the Euclidean or Minkowskian spaces, considered as mathematical models of space or space-time [6]. On this background, the generalizations of the notions of Coriolis' and centrifugal (centripetal) accelerations [7] from classical mechanics in Euclidean spaces are worth to be investigated in spaces with affine connections and metrics [$(\overline{L}_n, g)$-spaces] [8] $\div$ [12].

2. Usually, the Coriolis' and centrifugal (centripetal) accelerations are considered as apparent accelerations, generated by the non-inertial motion of the basic vector fields determining a co-ordinate system or a frame of reference in an Euclidean space. In Einstein's theory of gravitation (ETG) these types of accelerations are considered to be generated by a symmetric affine connection (Riemannian connection) compatible with the corresponding Riemannian metric [5]. In both cases they are considered as corollaries of the non-inertial motion of particles (i.e. of the motion of particles in non-inertial co-ordinate system).

3. In the present paper the notions of Coriolis' and centrifugal (centripetal) accelerations are considered with respect to their relations with the geometric characteristics of the corresponding models of space or space-time. It appears that accelerations of these types are closely related to the kinematic characteristics of the relative velocity and of the relative acceleration. The main idea is to be found out how a Coriolis' or centrifugal (centripetal) acceleration acts on a mass element of a flow or on a single particle during its motion in space or space-time described by a space of affine connections and metrics. In Section 1 the notions of centrifugal (centripetal) and Coriolis' velocities are introduced and considered in $(\overline{L}_n, g)$-spaces. The relations between the kinematic characteristics of the relative velocity and the introduced notions are established. The centrifugal (centripetal) velocity is found to be in connection with the Hubble law in spaces with affine connections and metrics. In Section 2 the notions of centrifugal (centripetal) and Coriolis' accelerations are introduced and their relations to the kinematic characteristics of the relative acceleration are found. In Section 3 the interpretation of the centrifugal (centripetal) acceleration as gravitational acceleration is given and illustrated on the basis of the Einstein theory of gravitation and especially on the basis of the Schwarzschild metric in vacuum.

4. The main results in the paper are given in details (even in full details) for these readers who are not familiar with the considered problems. The definitions and abbreviations are identical to those used in [9], [11], [12]. The reader is kindly asked to refer to them for more details and explanations of the statements and results only cited in this paper.

2 Centrifugal (centripetal) and Coriolis' velocities

Let us now recall some well known facts from kinematics of vector fields over spaces with affine connections and metrics [$(\overline{L}_n, g)$ - spaces], considered as models of space or space-time [1]÷[12].

1. Every contravariant vector field ξ could be represented by the use of a non-isotropic (non-null) contravariant vector field u and its corresponding contravariant and covariant projective metrics h^u and h_u in the form

$$\xi = \frac{1}{e} \cdot g(u, \xi) \cdot u + \overline{g}[h_u(\xi)] \ , \tag{1}$$

where

$$h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u \ , \qquad h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u) \ , \tag{2}$$

$$\overline{g} = g^{ij} \cdot e_i.e_j \ , \quad g^{ij} = g^{ji} \ , \quad e_i.e_j = \frac{1}{2} \cdot (e_i \otimes e_j + e_j \otimes e_i) \ ,$$

$$g = g_{ij} \cdot e^i.e^j \ , \quad g_{ij} = g_{ji} \ , \quad e^i.e^j = \frac{1}{2} \cdot (e^i \otimes e^j + e^j \otimes e^i) \ ,$$

$$e = g(u, u) = g_{\overline{kl}} \cdot u^k \cdot u^l = u_{\overline{k}} \cdot u^k = u_k \cdot u^{\overline{k}} :\neq 0 \ , \tag{3}$$

$$g(u, \xi) = g_{\overline{ij}} \cdot u^i \cdot \xi^j \ , \ e_i = \partial_i \ , \quad e^j = dx^j \ \text{ in a co-ordinate basis.} \tag{4}$$

By means of the representation of ξ the notions of relative velocity and relative acceleration are introduced [9], [11] in $(\overline{L}_n, g)$ - spaces. By that, the contravariant vector field ξ has been considered as vector field orthogonal to the contravariant vector field u, i.e. $g(u, \xi) = 0$, $\xi = \xi_\perp = \overline{g}[h_u(\xi)]$. Both the fields are considered as tangent vector fields to the corresponding co-ordinates, i.e. they fulfil the condition [8] $\pounds_\xi u = -\pounds_u \xi = [u, \xi] = 0$, where $[u, \xi] = u \circ \xi - \xi \circ u$. Under these preliminary conditions the relative velocity ${}_{rel}v$ could be found in the form [9], [11]

$${}_{rel}v = \overline{g}[d(\xi_\perp)] \ , \tag{5}$$

where

$$d = \sigma + \omega + \frac{1}{n-1} \cdot \theta \cdot h_u \ . \tag{6}$$

The covariant tensor d is the deformation velocity tensor; the covariant symmetric trace-free [$\overline{g}[\sigma] = g^{\overline{ij}} \cdot \sigma_{ij} = 0$] tensor σ is the shear velocity tensor; the covariant antisymmetric tensor ω is the rotation velocity tensor; the invariant θ is the expansion velocity invariant. The contravariant vector field ${}_{rel}v$ is interpreted as the relative velocity of two points (mass elements, particles) moving in a space or space-time and having equal proper times [1]÷[4]. The vector field $\xi_\perp$ is orthogonal to u and is interpreted as deviation vector connecting the two mass elements (particles) (if considered as an infinitesimal vector field).

2. Let us now consider the representation of the relative velocity by the use of a non-isotropic (non-null) contravariant vector field $\xi_\perp$ (orthogonal to u) and its corresponding projective metrics

$$h^{\xi_\perp} = \overline{g} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp \quad , \tag{7}$$

$$h_{\xi_\perp} = g - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp) \quad . \tag{8}$$

Then a contravariant vector field v could be represented in the form

$$v = \frac{g(v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \overline{g}[h_{\xi_\perp}(v)] \quad . \tag{9}$$

Therefore, the relative velocity ${}_{rel}v$ as a contravariant vector field could be now written in the form

$${}_{rel}v = \frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \overline{g}[h_{\xi_\perp}({}_{rel}v)] = v_z + v_c \quad , \tag{10}$$

where

$$v_z = \frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \quad , \qquad v_c = \overline{g}[h_{\xi_\perp}({}_{rel}v)] \quad . \tag{11}$$

The vector field v_z is collinear to the vector field $\xi_\perp$. If the factor (invariant) before $\xi_\perp$ is positive, i.e. if

$$\frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} > 0 \tag{12}$$

the vector field v_z is called (relative) *centrifugal velocity*. If the factor (invariant) before $\xi_\perp$ is negative, i.e. if

$$\frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} < 0 \tag{13}$$

the vector field v_z is called (relative) *centripetal velocity*.

The vector field v_c is called (relative) *Coriolis' velocity*.

Remark. The word (relative) is used here to stress the fact that the centripetal and the Coriolis velocities appear with respect to a trajectory interpreted as the world line of an observer with his sub space $T^{\perp u}(M)$, orthogonal to u at every point of his world line..

3. We can introduce the (normalized) contravariant unit vectors $n_\parallel$ and $n_\perp$ as

$$n_\parallel = \frac{1}{l_u} \cdot u \quad , \qquad n_\perp = \frac{1}{l_{\xi_\perp}} \cdot \xi_\perp \quad , \tag{14}$$

where

$$g(u,u) = e = \pm l_u^2 \quad , \qquad g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2 \quad , \tag{15}$$

$$g(n_\parallel, n_\parallel) = \frac{1}{l_u^2} \cdot g(u,u) = \pm \frac{l_u^2}{l_u^2} = \pm 1 \ , \tag{16}$$

$$g(n_\perp, n_\perp) = \frac{1}{l_{\xi_\perp}^2} \cdot g(\xi_\perp, \xi_\perp) = \mp \frac{l_{\xi_\perp}^2}{l_{\xi_\perp}^2} = \mp 1 \quad . \tag{17}$$

Remark. The signs $(\pm)$ and $(\mp)$ are depending on the signature of the metric g of the space or space-time. If $Sgn\, g > 0$ then $g(u,u) = -l_u^2$, $g(\xi_\perp, \xi_\perp) = +l_{\xi_\perp}^2$, $g(n_\parallel, n_\parallel) = -1$, $g(n_\perp, n_\perp) = +1$. If $Sgn\, g < 0$ then $g(u,u) = +l_u^2$, $g(\xi_\perp, \xi_\perp) = -l_{\xi_\perp}^2$, $g(n_\parallel, n_\parallel) = +1$, $g(n_\perp, n_\perp) = -1$.

By the use of the contravariant unit vectors $n_\parallel$ and $n_\perp$ the (relative) centrifugal (centripetal) velocity v_z could be written in the form

$$v_z = \mp g({}_{rel}v, n_\perp) \cdot n_\perp \quad . \tag{18}$$

2.1 Centrifugal (centripetal) velocity

Properties of the centrifugal (centripetal) velocity (a) Since v_z is collinear to $\xi_\perp$, it is orthogonal to the vector field u, i.e.

$$g(u, v_z) = 0 \quad . \tag{19}$$

(b) The centrifugal (centripetal) velocity v_z is orthogonal to the Coriolis velocity v_c

$$g(v_z, v_c) = 0 \ . \tag{20}$$

(c) The length of the vector v_z could be found by means of the relation

$$g(v_z, v_z) = v_z^2 = \mp l_{v_z}^2 = g(\frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp, \frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp) = \tag{21}$$

$$= \frac{[g({}_{rel}v, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)^2} \cdot g(\xi_\perp, \xi_\perp) = \frac{[g({}_{rel}v, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} = \frac{[g({}_{rel}v, \xi_\perp)]^2}{\xi_\perp^2} =$$

$$= \mp \frac{[g({}_{rel}v, \xi_\perp)]^2}{l_{\xi_\perp}^2} = \mp \frac{l_{\xi_\perp}^2 \cdot [g({}_{rel}v, n_\perp)]^2}{l_{\xi_\perp}^2} = \mp [g({}_{rel}v, n_\perp)]^2 \ . \tag{22}$$

From the last (previous) expression we can conclude that the square of the length of the centrifugal (centripetal) velocity is in general in inverse proportion to the length of the vector field $\xi_\perp$.

Special case: $M_n := E_n$, $n = 3$ (3-dimensional Euclidean space): $\xi_\perp = \overrightarrow{r}$, $g(\xi_\perp, \xi_\perp) = r^2$

$$v_z^2 = \frac{[g(\overrightarrow{r}, {}_{rel}\overrightarrow{v})]^2}{r^2} \ , \qquad l_{v_z} = \sqrt{v_z^2} = \frac{g(\overrightarrow{r}, {}_{rel}\overrightarrow{v})}{r} \quad . \tag{23}$$

If the relative velocity ${}_{rel}v$ is equal to zero then $v_z = 0$.

(d) The scalar product between v_z and ${}_{rel}v$ could be found in its explicit form by the use of the explicit form of the relative velocity ${}_{rel}v$

$$ {}_{rel}v = \overline{g}[d(\xi_\perp)] = \overline{g}[\sigma(\xi_\perp)] + \overline{g}[\omega(\xi_\perp)] + \frac{1}{n-1} \cdot \theta \cdot \xi_\perp \ , \tag{24} $$

and the relations

$$ g(\xi_\perp, {}_{rel}v) = g(\xi_\perp, \overline{g}[\sigma(\xi_\perp)]) + \overline{g}(\xi_\perp, \overline{g}[\omega(\xi_\perp)]) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp) \ , \tag{25} $$

$$ g(\xi_\perp, \overline{g}[\sigma(\xi_\perp)]) = \sigma(\xi_\perp, \xi_\perp) \ , \tag{26} $$

$$ \overline{g}(\xi_\perp, \overline{g}[\omega(\xi_\perp)]) = \omega(\xi_\perp, \xi_\perp) = 0 \ , \tag{27} $$

$$ g(\xi_\perp, {}_{rel}v) = \sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp) \ . \tag{28} $$

For

$$ \begin{aligned} g(v_z, {}_{rel}v) &= g(\frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} . \xi_\perp, {}_{rel}v) = \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp, {}_{rel}v) = \\ &= \frac{[g(\xi_\perp, {}_{rel}v)]^2}{g(\xi_\perp, \xi_\perp)} = v_z^2 \ , \end{aligned} \tag{29} $$

it follows that

$$ g(v_z, {}_{rel}v) = g(v_z, v_z) = v_z^2 \ . \tag{30} $$

Remark. $g(v_z, {}_{rel}v) = g(v_z, v_z) = v_z^2$ because of $g(v_z, v_c) = 0$ and $g(v_z, {}_{rel}v) = g(v_z, v_z + v_c) = g(v_z, v_z) + g(v_z, v_c) = g(v_z, v_z)$.

On the other side, we can express v_z^2 by the use of the kinematic characteristics of the relative velocity σ, ω, and θ

$$ \begin{aligned} v_z^2 &= \frac{[g(\xi_\perp, {}_{rel}v)]^2}{g(\xi_\perp, \xi_\perp)} = \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot [\sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp)]^2 = \\ &= \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \{[\sigma(\xi_\perp, \xi_\perp)]^2 + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot [g(\xi_\perp, \xi_\perp)]^2 + \\ &+ \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp) \cdot g(\xi_\perp, \xi_\perp)\} \ , \\ v_z^2 &= \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) + \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp) \ , \end{aligned} \tag{31} $$

$$ \begin{aligned} v_z^2 &= \mp l_{v_z}^2 = \mp l_{\xi_\perp}^2 \cdot [\sigma(n_\perp, n_\perp)]^2 \mp \frac{1}{(n-1)^2} \cdot \theta^2 \cdot l_{\xi_\perp}^2 + \frac{2}{n-1} \cdot \theta \cdot \xi_\perp^2 \cdot \sigma(n_\perp, n_\perp) = \\ &= \mp l_{\xi_\perp}^2 \cdot \{[\sigma(n_\perp, n_\perp)]^2 + \frac{1}{(n-1)^2} \cdot \theta^2 \mp \frac{2}{n-1} \cdot \theta \cdot \sigma(n_\perp, n_\perp)\} = \\ &= \mp l_{\xi_\perp}^2 \cdot [\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta]^2 \quad , \end{aligned} $$

$$l_{v_z}^2 = l_{\xi_\perp}^2 \cdot [\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta]^2 = [\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)]^2 \cdot l_{\xi_\perp}^2 \quad . \tag{32}$$

Therefore, for l_{v_z} we obtain

$$l_{v_z} = \pm[\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta] \cdot l_{\xi_\perp} = \tag{33}$$

$$= \ [\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)] \cdot l_{\xi_\perp} \quad . \tag{34}$$

Special case: $\theta := 0$ (expansion-free relative velocity)

$$v_z^2 = \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} = \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{\xi_\perp^2} = \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{\mp l_{\xi_\perp}^2} \ , \tag{35}$$

$$g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2 \quad , \tag{36}$$

$$g(\xi_\perp, \xi_\perp) = -l_{\xi_\perp}^2 \quad \text{if} \quad g(u,u) = e = +l_u^2 \quad , \tag{37}$$

$$g(\xi_\perp, \xi_\perp) = +l_{\xi_\perp}^2 \quad \text{if} \quad g(u,u) = e = -l_u^2 \quad . \tag{38}$$

Remark. If we introduce as above the unit vector field $n_\perp$, normal to u and normalized, as

$$n_\perp := \frac{\xi_\perp}{l_{\xi_\perp}} \quad , \qquad g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2 \quad , \quad \xi_\perp = l_{\xi_\perp} \cdot n_\perp \quad , \tag{39}$$

$$l_{\xi_\perp} = \mid g(\xi_\perp, \xi_\perp) \mid^{1/2} \quad , \quad g(n_\perp, n_\perp) = \mp 1 \quad . \tag{40}$$

then the above expressions for v_z^2 with $[\sigma(\xi_\perp, \xi_\perp)]^2 / \ g(\xi_\perp, \xi_\perp)$ could be written in the form:

$$v_z^2 = \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{\mp l_{\xi_\perp}^2} = \mp l_{\xi_\perp}^2 \cdot [\sigma(n_\perp, n_\perp)]^2 \quad . \tag{41}$$

Special case: $\sigma := 0$ (shear-free relative velocity)

$$v_z^2 = \mp l_{v_z}^2 = \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) = \mp \frac{1}{(n-1)^2} \cdot \theta^2 \cdot l_{\xi_\perp}^2 \ , \tag{42}$$

$$l_{v_z}^2 = \frac{1}{(n-1)^2} \cdot \theta^2 \cdot l_{\xi_\perp}^2 \quad . \tag{43}$$

(e) The explicit form of v_z could be found as

$$\begin{aligned} v_z &= \frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot [\sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp)] \cdot \xi_\perp = \\ &= [\frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} + \frac{1}{n-1} \cdot \theta] \cdot \xi_\perp = [\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)] \cdot \xi_\perp \ , \\ v_z &= [\frac{1}{n-1} \cdot \theta + \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)}] \cdot \xi_\perp = [\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)] \cdot l_{\xi_\perp} \cdot n_\perp \ . \end{aligned} \tag{44}$$

If

$$\frac{1}{n-1} \cdot \theta + \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} > 0 \ , \tag{45}$$

$$\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) > 0 \quad , \qquad \xi_\perp^2 = g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2 \ , \tag{46}$$

we have a centrifugal (relative) velocity.

If

$$\frac{1}{n-1} \cdot \theta + \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} < 0 \ , \tag{47}$$

$$\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) < 0 \ , \tag{48}$$

we have a centripetal (relative) velocity.

Special case: $\theta := 0$ (expansion-free relative velocity)

$$v_z = \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \mp \sigma(n_\perp, n_\perp) \cdot l_{\xi_\perp} \cdot n_\perp \ . \tag{49}$$

Special case: $\sigma := 0$ (shear-free relative velocity)

$$v_z = \frac{1}{n-1} \cdot \theta \cdot \xi_\perp = \frac{1}{n-1} \cdot \theta \cdot l_{\xi_\perp} \cdot n_\perp \quad . \tag{50}$$

If the expansion invariant $\theta > 0$ we have centrifugal (or expansion) (relative) velocity. If the expansion invariant $\theta < 0$ we have centripetal (or contraction) (relative) velocity. Therefore, in the case of a shear-free relative velocity v_z is proportional to the expansion velocity invariant θ.

The centrifugal (centripetal) relative velocity v_z could be also written in the form

$$\begin{aligned} v_z &= \frac{g({}_{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \mp \frac{g({}_{rel}v, \xi_\perp)}{l_{\xi_\perp}^2} \cdot \xi_\perp = \\ &= \mp g({}_{rel}v, \frac{\xi_\perp}{l_{\xi_\perp}}) \cdot \frac{\xi_\perp}{l_{\xi_\perp}} = \mp g({}_{rel}v, n_\perp) \cdot n_\perp \ . \end{aligned} \tag{51}$$

On the other side, on the basis of the relations

$$\begin{aligned} {}_{rel}v &= \overline{g}[d(\xi_\perp)] = \overline{g}[\sigma(\xi_\perp)] + \overline{g}[\omega(\xi_\perp)] + \frac{1}{n-1} \cdot \theta \cdot \xi_\perp \ , \\ \overline{g}[h_u(\xi)] &= \xi_\perp \quad , \quad \overline{g}[h_u(\xi_\perp)] = \overline{g}[h_u(\overline{g}[h_u(\xi)])] = \overline{g}[h_u(\xi)] = \xi_\perp \ , \\ g(\overline{g}[\sigma(\xi_\perp)], n_\perp) &= \sigma(n_\perp, \xi_\perp) = \sigma(n_\perp, l_{\xi_\perp} \cdot n_\perp) = l_{\xi_\perp} \cdot \sigma(n_\perp, n_\perp) \ , \\ g(\overline{g}[\omega(\xi_\perp)], n_\perp) &= \omega(n_\perp, \xi_\perp) = \omega(n_\perp, l_{\xi_\perp} \cdot n_\perp) = l_{\xi_\perp} \cdot \omega(n_\perp, n_\perp) = 0 \quad , \\ g(\xi_\perp, n_\perp) &= l_{\xi_\perp} \cdot g(n_\perp, n_\perp) = \mp l_{\xi_\perp} \ , \end{aligned}$$

it follows for $g({}_{rel}v, n_\perp)$

$$g({}_{rel}v, n_\perp) = [\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta] \cdot l_{\xi_\perp} \quad , \tag{52}$$

and for v_z

$$v_z = \mp g({}_{rel}v, n_\perp) \cdot n_\perp = [\mp\sigma(n_\perp, n_\perp) + \frac{1}{n-1} \cdot \theta] \cdot l_{\xi_\perp} \cdot n_\perp \quad ,$$
$$v_z = [\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)] \cdot l_{\xi_\perp} \cdot n_\perp \quad . \tag{53}$$

Since $l_{\xi_\perp} > 0$, we have three different cases:
(a)

$$v_z > 0 : \mp\sigma(n_\perp, n_\perp) + \frac{1}{n-1} \cdot \theta > 0 :$$
$$\theta > \pm(n-1) \cdot \sigma(n_\perp, n_\perp) \ , \qquad n-1 > 0 \ .$$

(b)

$$v_z < 0 : \mp\sigma(n_\perp, n_\perp) + \frac{1}{n-1} \cdot \theta < 0 :$$
$$\theta < \pm(n-1) \cdot \sigma(n_\perp, n_\perp) \ , \qquad n-1 > 0 \ .$$

(c)

$$v_z = 0 : \mp\sigma(n_\perp, n_\perp) + \frac{1}{n-1} \cdot \theta = 0 :$$
$$\theta = \pm(n-1) \cdot \sigma(n_\perp, n_\perp) \ , \qquad n-1 > 0 \quad .$$

Special case: $\sigma := 0$ (shear-free relative velocity)

$$v_z = \frac{1}{n-1} \cdot \theta \cdot l_{\xi_\perp} \cdot n_\perp \quad .$$

For $v_z > 0 : \theta > 0$ we have an expansion, for $v_z < 0 : \theta < 0$ we have a contraction, and for $v_z = 0 : \theta = 0$ we have a stationary case.

Special case: $\theta := 0$ (expansion-free relative velocity)

$$v_z = \mp\sigma(n_\perp, n_\perp) \cdot l_{\xi_\perp} \cdot n_\perp \quad .$$

For $v_z > 0 : \mp\sigma(n_\perp, n_\perp) > 0$ we have an expansion, for $v_z < 0 : \mp\sigma(n_\perp, n_\perp) < 0$ we have a contraction, and for $v_z = 0 : \sigma(n_\perp, n_\perp) = 0$ we have a stationary case.

In an analogous way we can find the explicit form of v_z^2 as

$$v_z^2 = \{\mp[\sigma(n_\perp, n_\perp)]^2 \mp \frac{1}{(n-1)^2} \cdot \theta^2 + \frac{2}{n-1} \cdot \theta \cdot \sigma(n_\perp, n_\perp)\} \cdot l_{\xi_\perp}^2 = \tag{54}$$
$$= \mp[\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta]^2 \cdot l_{\xi_\perp}^2 \ . \tag{55}$$

The expression for v_z^2 could be now written in the form

$$v_z^2 = \mp H^2 \cdot l_{\xi_\perp}^2 = \mp l_{v_z}^2 \quad , \tag{56}$$

where

$$H^2 = [\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta]^2 \ . \tag{57}$$

Then

$$l_{v_z}^2 = H^2 \cdot l_{\xi_\perp}^2 \quad , \qquad l_{v_z} = \mid H \mid \cdot l_{\xi_\perp} \quad , \quad l_{v_z} > 0 \ \ , \ \ l_{\xi_\perp} > 0 \ \ , \tag{58}$$

$$H = \mp[\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta] = \tag{59}$$

$$= \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \quad . \tag{60}$$

and

$$v_z = \pm\, l_{v_z} \cdot n_\perp = \pm \mid H \mid \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp \quad . \tag{61}$$

Remark. From the forms of $v_z = \pm l_{v_z} \cdot n_\perp$ and $v_z = [\frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)] \cdot l_{\xi_\perp} \cdot n_\perp$ it follows immediately that

$$\pm l_{v_z} = H \cdot l_{\xi_\perp} \quad , \qquad v_z = H \cdot l_{\xi_\perp} \cdot n_\perp \quad . \tag{62}$$

It follows from the last (previously) expression that the centrifugal (centripetal) relative velocity $v_z = H \cdot \xi_\perp$ is collinear to the vector field $\xi_\perp$.

The absolute value $l_{v_z} = \mid H \mid \cdot l_{\xi_\perp}$ of v_z and the expression for the centrifugal (centripetal) relative velocity $v_z = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp$ represent generalizations of the *Hubble law* [15], [5]. The function H could be called *Hubble function* (some authors call it *Hubble coefficient* [15], [5]). Since $H = H(x^k(\tau)) = H(\tau)$, for a given proper time $\tau = \tau_0$ the function H has at the time τ_0 the value $H(\tau_0) = H(\tau = \tau_0) =$ const. The Hubble function H is usually called *Hubble constant.*

Remark. The Hubble coefficient H has dimension sec^{-1}. The function H^{-1} with dimension *sec* is usually denoted in astrophysics as *Hubble's time* [15].

The Hubble function H could also be represented in the forms [12]

$$H = \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) =$$

$$= \frac{1}{2}[(n_\perp)(h_u)(\nabla_u \overline{g} - \pounds_u \overline{g})(h_u)(n_\perp)] \quad . \tag{63}$$

In the Einstein theory of gravitation (ETG) the Hubble coefficient is considered under the condition that the centrifugal relative velocity is generated by a shear-free relative velocity in a cosmological model of the type of Robertson-Walker [15], [5], i.e.

$$v_z = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp \quad \text{with} \qquad \sigma = 0 \quad . \tag{64}$$

From the explicit form of H, it follows that in this case

$$H = \frac{1}{n-1} \cdot \theta$$

and the Hubble function H depends only on the expansion velocity θ.

Special case: U_n - or V_n-space: $\nabla_u \overline{g} = 0$, $\pounds_u \overline{g} := 0$ (the contravariant vector field u is a Killing vector field in the corresponding space)

$$H = 0 \ \ .$$

Therefore, in a (pseudo) Riemannian space with or without torsion (U_n- or V_n- space) the Hubble function H is equal to zero if the velocity vector field u ($n = 4$) of an observer is a Killing vector field fulfilling the Killing equation $\pounds_u \overline{g} = 0$. This means that the condition $\pounds_u \overline{g} = 0$ is a sufficient condition for the Hubble function H to be equal to zero. Therefore, an observer, moving in space-time with a velocity vector field u obeying the condition $\pounds_v g = 0$, could not detect any centrifugal (centripetal) relative velocity v_z. All mass elements (particles) in the surroundings of the observer could only circle around him (without any expansion or contraction, related to the relative centrifugal (centripetal) velocity) and will have stable orbits with respect to this observer.

Special case: $(\overline{L}_n, g)$-space with $\nabla_u \overline{g} - \pounds_u \overline{g} := 0$ [16]. For this case, the last expression appears as a sufficient condition for the vanishing of the Hubble function H, i.e. $H = 0$ if $\nabla_u \overline{g} = \pounds_u \overline{g}$. This is an analogous case for a $(\overline{L}_n, g)$-space as the case in U_n- and V_n-spaces, where no centrifugal (centripetal) velocity could be detected.

Special case: $(\overline{L}_n, g)$-spaces with vanishing centrifugal (centripetal) velocity: $v_z := 0$.

$$\begin{aligned} & v_z := 0 : H = 0 : \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) = 0 \quad , \\ & \theta = \pm(n-1) \cdot \sigma(n_\perp, n_\perp) \quad . \end{aligned}$$

Therefore, if a $(\overline{L}_n, g)$-space allows the existence of an observer, who could not detect any centrifugal (centripetal) relative velocity, the expansion velocity is compensated by the shear velocity in the space on the basis of the relation $\theta = \pm(n-1) \cdot \sigma(n_\perp, n_\perp)$.

Remark. The Hubble function H is introduced in the above considerations only on a purely kinematic basis related to the notions of relative velocity and centrifugal (centripetal) relative velocity. Its dynamic interpretation in a theory of gravitation depends on the structure of the theory and the relations between the field equations and the Hubble function.

2.2 Coriolis' velocity

The vector field

$$v_c = \overline{g}[h_{\xi_\perp}({}_{rel}v)] = g^{ij} \cdot (h_{\xi_\perp})_{\overline{j}\overline{k}} \cdot \ {}_{rel}v^k \cdot \partial_i \tag{65}$$

is called Coriolis' velocity.

Properties of the Coriolis' velocity (a) The Coriolis velocity is orthogonal to the vector field u, interpreted as velocity of a mass element (particle), i.e.

$$g(u, v_c) = 0 \ . \tag{66}$$

Proof: From the definition of the Coriolis velocity, it follows

$$\begin{aligned} g(u, v_c) &= g(u, \overline{g}[h_{\xi_\perp}({}_{rel}v)] = g_{\overline{ij}} \cdot u^i \cdot g^{jk} \cdot (h_{\xi_\perp})_{\overline{kl}} \cdot \ {}_{rel}v^l = \\ &= g_{\overline{ij}} \cdot g^{jk} \cdot u^i \cdot (h_{\xi_\perp})_{\overline{kl}} \cdot \ {}_{rel}v^l = g_i^k \cdot u^i \cdot (h_{\xi_\perp})_{\overline{kl}} \cdot \ {}_{rel}v^l = \\ &= (h_{\xi_\perp})_{\overline{kl}} \cdot u^k \cdot \ {}_{rel}v^l = (u)(h_{\xi_\perp})({}_{rel}v) \ . \end{aligned} \tag{67}$$

Since

$$\begin{aligned} (u)(h_{\xi_\perp}) &= (h_{\xi_\perp})_{\overline{kl}} \cdot u^k \cdot dx^l = (u)(g) - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot (u)[g(\xi_\perp)] \cdot g(\xi_\perp) = \\ &= g(u) - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(u, \xi_\perp) \cdot g(\xi_\perp) = g(u) \quad , \end{aligned} \tag{68}$$

$$g(u, \xi_\perp) = 0 \ \ ,$$

then $(u)(h_{\xi_\perp})({}_{rel}v) = [g(u)]({}_{rel}v) = g(u, {}_{rel}v) = 0$. Because of $g(u, {}_{rel}v) = 0$, it follows that $g(u, v_c) = 0$.

(b) The Coriolis velocity v_c is orthogonal to the centrifugal (centripetal) velocity v_z

$$g(v_c, v_z) = 0 \ . \tag{69}$$

(c) The length of the vector v_c could be found by the use of the relations:

$$v_c = \overline{g}[h_{\xi_\perp}({}_{rel}v)] \qquad ,$$

$$\begin{aligned} h_{\xi_\perp}({}_{rel}v) &= g({}_{rel}v) - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot [g(\xi_\perp)]({}_{rel}v) \cdot g(\xi_\perp) = \\ &= g({}_{rel}v) - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) = \qquad (70) \\ &= g({}_{rel}v) \pm g(n_\perp, \ {}_{rel}v) \cdot g(n_\perp) \ \ , \qquad (71) \end{aligned}$$

$$[g(\xi_\perp)]({}_{rel}v) = g(\xi_\perp, {}_{rel}v) \ ,$$

$$\begin{aligned} v_c = \overline{g}[h_{\xi_\perp}({}_{rel}v)] &= \ {}_{rel}v - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \qquad (72) \\ &= {}_{rel}v \ \pm g(n_\perp, \ {}_{rel}v) \cdot n_\perp = \ {}_{rel}v - v_z \quad , \qquad (73) \end{aligned}$$

$$\begin{aligned} g(v_c, v_c) = v_c^2 = \mp l_{v_c}^2 &= g({}_{rel}v - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp, {}_{rel}v - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp) = \\ &= g({}_{rel}v, {}_{rel}v) - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp, {}_{rel}v) - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot g({}_{rel}v, \xi_\perp) + \\ &+ \frac{[g(\xi_\perp, {}_{rel}v)]^2}{g(\xi_\perp, \xi_\perp)} \ , \end{aligned}$$

$$g(v_c, v_c) = v_c^2 = \mp l_{v_c}^2 = g({}_{rel}v, {}_{rel}v) - \frac{[g(\xi_\perp, {}_{rel}v)]^2}{g(\xi_\perp, \xi_\perp)} \quad . \tag{74}$$

On the other side, because of $(u)(h_u) = h_u(u) = 0$, and ${}_{rel}v = \overline{g}[h_u(\nabla_u \xi_\perp)]$, it follows that

$$g(u, {}_{rel}v) = h_u(u, \nabla_u \xi_\perp) = (u)(h_u)(\nabla_u \xi_\perp) \quad ,$$

$$g(u, {}_{rel}v) = (u)(h_u)(\nabla_u \xi_\perp) = 0 \ . \tag{75}$$

Since

$$\begin{aligned} g({}_{rel}v, {}_{rel}v) = {}_{rel}v^2 = \mp l_{{}_{rel}v}^2 = g(\overline{g}[d(\xi_\perp)], \overline{g}[d(\xi_\perp)]) = \\ = \overline{g}(d(\xi_\perp), d(\xi_\perp)) \quad , \quad \pounds_u \xi_\perp = 0 \quad , \end{aligned} \tag{76}$$

$$\begin{aligned} \overline{g}(d(\xi_\perp), d(\xi_\perp)) = \overline{g}(\sigma(\xi_\perp) + \omega(\xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp), \\ \sigma(\xi_\perp) + \omega(\xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp)) \end{aligned}$$

we obtain

$$\begin{aligned} g({}_{rel}v, {}_{rel}v) = {}_{rel}v^2 = \overline{g}(d(\xi_\perp), d(\xi_\perp)) = \\ = \overline{g}(\sigma(\xi_\perp), \sigma(\xi_\perp)) + \overline{g}(\omega(\xi_\perp), \omega(\xi_\perp)) + \\ + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) + \\ + 2 \cdot \overline{g}(\sigma(\xi_\perp), \omega(\xi_\perp)) + \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp) \ . \end{aligned} \tag{77}$$

Special case: $\sigma := 0$, $\theta := 0$ (shear-free and expansion-free velocity).

$${}_{rel}v^2 = \overline{g}(\omega(\xi_\perp), \omega(\xi_\perp)) \ . \tag{78}$$

Special case: $\omega := 0$ (rotation-free velocity).

$${}_{rel}v^2 = \overline{g}(\sigma(\xi_\perp), \sigma(\xi_\perp)) + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) + \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp) \ . \tag{79}$$

Special case: $\theta := 0$, $\omega := 0$ (expansion-free and rotation-free velocity).

$${}_{rel}v^2 = \overline{g}(\sigma(\xi_\perp), \sigma(\xi_\perp)) \quad . \tag{80}$$

Special case: $\sigma := 0$, $\omega := 0$ (shear-free and rotation-free velocity).

$${}_{rel}v^2 = \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) \quad . \tag{81}$$

Special case: $\theta := 0$ (expansion-free velocity).

$$_{rel}v^2 = \overline{g}(\sigma(\xi_\perp), \sigma(\xi_\perp)) + \overline{g}(\omega(\xi_\perp), \omega(\xi_\perp)) + 2 \cdot \overline{g}(\sigma(\xi_\perp), \omega(\xi_\perp)) \ . \tag{82}$$

The square v_c^2 of v_c could be found on the basis of the relation $v_c^2 = \ _{rel}v^2 - v_z^2$

$$\begin{aligned} v_c^2 = \ _{rel}v^2 - v_z^2 = \overline{g}(\sigma(\xi_\perp), \sigma(\xi_\perp)) - \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} + \\ + \overline{g}(\omega(\xi_\perp), \omega(\xi_\perp)) + 2 \cdot \overline{g}(\sigma(\xi_\perp), \omega(\xi_\perp)) \ . \end{aligned} \tag{83}$$

Therefore, the length of v_c does not depend on the expansion velocity invariant θ.

Special case: $\sigma := 0$ (shear-free velocity).

$$v_c^2 = \overline{g}(\omega(\xi_\perp), \omega(\xi_\perp)) \ . \tag{84}$$

Special case: $\omega := 0$ (rotation-free velocity).

$$v_c^2 = \overline{g}(\sigma(\xi_\perp), \sigma(\xi_\perp)) - \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} \ . \tag{85}$$

It follows from the last (previous) expression that even if the rotation velocity tensor ω is equal to zero ($\omega = 0$), the Coriolis (relative) velocity v_c is not equal to zero if $\sigma \neq 0$.

(d) The scalar product between v_c and $_{rel}v$ could be found in its explicit form by the use of the relations:

$$g(v_c, {}_{rel}v) = h_{\xi_\perp}({}_{rel}v, {}_{rel}v) = \ _{rel}v^2 - \frac{[g(\xi_\perp, {}_{rel}v)]^2}{g(\xi_\perp, \xi_\perp)} = \ _{rel}v^2 - v_z^2 = v_c^2 \ . \tag{86}$$

(e) The explicit form of v_c could be found by the use of the relations

$$v_c = \overline{g}[h_{\xi_\perp}({}_{rel}v)] = \ _{rel}v - \frac{g(\xi_\perp, {}_{rel}v)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \ _{rel}v - v_z \ ,$$

$$v_c = \overline{g}[\sigma(\xi_\perp)] - \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \overline{g}[\omega(\xi_\perp)] \ . \tag{87}$$

Therefore, the Coriolis velocity does not depend on the expansion velocity invariant θ.

Special case: $\sigma := 0$ (shear-free velocity).

$$v_c = \overline{g}[\omega(\xi_\perp)] \ . \tag{88}$$

Special case: $\omega := 0$ (rotation-free velocity).

$$v_c = \overline{g}[\sigma(\xi_\perp)] - \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \ . \tag{89}$$

(f) The Coriolis velocity v_c is orthogonal to the deviation vector $\xi_\perp$, i.e. $g(v_c, \xi_\perp) = 0$.

Proof: From $g(v_c, \xi_\perp) = g(\overline{g}[h_{\xi_\perp}({}_{rel}v)], \xi_\perp) = h_{\xi_\perp}(\xi_\perp, {}_{rel}v) = (\xi_\perp)(h_{\xi_\perp})({}_{rel}v)$ and $(\xi_\perp)(h_{\xi_\perp}) = 0$, it follows that

$$g(v_c, \xi_\perp) = 0. \tag{90}$$

3 Centrifugal (centripetal) and Coriolis' accelerations

In analogous way as in the case of centrifugal (centripetal) and Coriolis' velocities, the corresponding accelerations could be defined by the use of the projections of the relative acceleration ${}_{rel}a = \overline{g}[h_u(\nabla_u \nabla_u \xi_\perp)]$ along or orthogonal to the vector field $\xi_\perp$

$$ {}_{rel}a = \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \overline{g}[h_{\xi_\perp}({}_{rel}a)] = a_z + a_c \quad , \tag{91} $$

where

$$ a_z = \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \quad , \tag{92} $$

$$ a_c = \overline{g}[h_{\xi_\perp}({}_{rel}a)] \quad . \tag{93} $$

If

$$ \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} > 0 \tag{94} $$

the vector field a_z is called (relative) *centrifugal acceleration.* If

$$ \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} < 0 \tag{95} $$

the vector field a_z is called (relative) *centripetal acceleration.*

The vector field a_c is called (relative) *Coriolis' acceleration.*

3.1 Centrifugal (centripetal) acceleration

The centrifugal (centripetal) acceleration has specific properties which will be considered below on the basis of the properties of the relative acceleration ${}_{rel}a$.

The relative acceleration ${}_{rel}a$ is orthogonal to the vector field u.

Proof: From $g(u, {}_{rel}a) = g(u, \overline{g}[h_u(\nabla_u \nabla_u \xi_\perp)]) = (u)(h_u)(\nabla_u \nabla_u \xi_\perp)$ and $(u)(h_u) = h_u(u) = 0$, it follows that

$$ g(u, {}_{rel}a) = 0 \ . \tag{96} $$

(a) The centrifugal (centripetal) acceleration a_z is orthogonal to the vector field u, i.e.

$$ g(u, a_z) = 0. \tag{97} $$

Proof: From

$$ \begin{aligned} g(u, a_z) &= g(u, \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp) = \\ &= \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot g(u, \xi_\perp) \ , \end{aligned} \tag{98} $$

$$ g(u, \xi_\perp) = 0 \ , $$

it follows that $g(u, a_z) = 0$.

(b) The centrifugal (centripetal) acceleration a_z is orthogonal to the Coriolis acceleration a_c, i.e.

$$g(a_z, a_c) = 0 \ . \tag{99}$$

Proof: From

$$g(\frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp, \overline{g}[h_{\xi_\perp}({}_{rel}a)]) = \frac{g(\xi_\perp, \ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp, \overline{g}[h_{\xi_\perp}({}_{rel}a)]) \ ,$$

$$g(\xi_\perp, \overline{g}[h_{\xi_\perp}({}_{rel}a)]) = (\xi_\perp)(h_{\xi_\perp})({}_{rel}a) \ , \tag{100}$$

$$(\xi_\perp)(h_{\xi_\perp}) = (h_{\xi_\perp})(\xi_\perp) = 0 \ , \tag{101}$$

it follows that $g(a_z, a_c) = 0$.

(c) The length of the vector a_z could be found on the basis of the relations

$$a_z^2 = g(a_z, a_z) = \mp l_{a_z}^2 = \frac{[g(\xi_\perp, \ {}_{rel}a)]^2}{g(\xi_\perp, \xi_\perp)} \quad . \tag{102}$$

Therefore, in general, the square a_z^2 of the length of the centrifugal (centripetal) acceleration a_z is reverse proportional to $\xi_\perp^2 = g(\xi_\perp, \xi_\perp)$.

Special case: $M_n = E_n$, $n = 3$ (3-dimensional Euclidean space).

$$\xi_\perp := \overrightarrow{r} \ , \quad g(\xi_\perp, \xi_\perp) = r^2 \quad , \quad {}_{rel}a = \ {}_{rel}\overrightarrow{a} \ ,$$

$$a_z^2 = \frac{[g(\overrightarrow{r}, \ {}_{rel}\overrightarrow{a})]^2}{r^2} \ , \qquad l_{a_z} = \frac{g(\overrightarrow{r}, {}_{rel}\overrightarrow{a})}{r} = g(n_\perp, \ {}_{rel}\overrightarrow{a}) \ , \tag{103}$$

$$n_\perp := \frac{\overrightarrow{r}}{r} \ , \qquad g(n_\perp, n_\perp) = \frac{g(\overrightarrow{r}, \overrightarrow{r})}{r^2} = n_\perp^2 = 1 \quad . \tag{104}$$

The length $l_{a_z} = \mid g(a_z, a_z) \mid^{1/2}$ of the centrifugal (centripetal) acceleration a_z is equal to the projection of the relative acceleration ${}_{rel}\overrightarrow{a}$ at the unit vector field $n_\perp$ along the vector field $\xi_\perp$. If $l_{a_z} = g(n_\perp, \ {}_{rel}\overrightarrow{a}) = 0$, i.e. if the relative acceleration ${}_{rel}\overrightarrow{a}$ is orthogonal to the radius vector $\overrightarrow{r}$ $[{}_{rel}\overrightarrow{a} \perp \overrightarrow{r}]$, then $a_z = 0$.

If the relative acceleration ${}_{rel}a$ is equal to zero then the centrifugal (centripetal) acceleration a_z is also equal to zero.

(d) The scalar product $g(\xi_\perp, {}_{rel}a)$ could be found in its explicit form by the use of the explicit form of ${}_{rel}a$ [9]

$${}_{rel}a = \overline{g}[A(\xi_\perp)] = \overline{g}[{}_sD(\xi_\perp)] + \overline{g}[W(\xi_\perp)] + \frac{1}{n-1} \cdot U \cdot \xi_\perp \ . \tag{105}$$

Then

$$g(\xi_\perp,\ {}_{rel}a) = \ {}_sD(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot U \cdot g(\xi_\perp, \xi_\perp) \ , \tag{106}$$

$$\begin{aligned} a_z^2 &= \frac{[g(\xi_\perp,\ {}_{rel}a)]^2}{g(\xi_\perp, \xi_\perp)} = \\ &= \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot [{}_sD(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot U \cdot g(\xi_\perp, \xi_\perp)]^2 = \\ &= \frac{[{}_sD(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} + \frac{1}{(n-1)^2} \cdot U^2 \cdot g(\xi_\perp, \xi_\perp) + \\ &+ \frac{2}{n-1} \cdot U \cdot \ {}_sD(\xi_\perp, \xi_\perp) \ . \end{aligned} \tag{107}$$

Special case: ${}_sD := 0$ (shear-free acceleration).

$$a_z^2 = \frac{1}{(n-1)^2} \cdot U^2 \cdot g(\xi_\perp, \xi_\perp) \quad . \tag{108}$$

Special case: $U := 0$ (expansion-free acceleration).

$$a_z^2 = \frac{[{}_sD(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} \quad . \tag{109}$$

(e) The explicit form of a_z could be found in the form

$$\begin{aligned} a_z &= \frac{g(\xi_\perp,\ {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \frac{1}{n-1} \cdot U \cdot \xi_\perp = \\ &= [\frac{1}{n-1} \cdot U + \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)}] \cdot \xi_\perp = \end{aligned} \tag{110}$$

$$= [\frac{1}{n-1} \cdot U \pm \ {}_sD(n_\perp, n_\perp)] \cdot \xi_\perp \quad . \tag{111}$$

If

$$\frac{1}{n-1} \cdot U + \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} > 0 \tag{112}$$

a_z is a centrifugal (relative) acceleration. If

$$\frac{1}{n-1} \cdot U + \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} < 0 \tag{113}$$

a_z is a centripetal (relative) acceleration.

The (relative) centrifugal (centripetal) acceleration a_z could be written also in the form

$$a_z = \overline{q} \cdot \xi_\perp \quad , \tag{114}$$

where

$$\overline{q} = \frac{1}{n-1} \cdot U \pm \ {}_sD(n_\perp, n_\perp) \ . \tag{115}$$

The invariant function $\overline{q}$ could be denoted as *acceleration function* or *acceleration parameter*. Its explicit form depends on the explicit forms of the expansion acceleration invariant U and on the shear acceleration tensor ${}_sD$ [9], [11]. The centrifugal (centripetal) acceleration a_z has an analogous form to the centrifugal (centripetal) velocity v_z

$$a_z = \overline{q} \cdot l_{\xi_\perp} \cdot n_\perp = \pm l_{a_z} \cdot n_\perp \quad , \tag{116}$$

$$v_z = H \cdot l_{\xi_\perp} \cdot n_\perp = \pm l_{v_z} \cdot n_\perp \quad . \tag{117}$$

Since $\pm l_{v_z} = H \cdot l_{\xi_\perp}$ and $\pm l_{a_z} = \overline{q} \cdot l_{\xi_\perp}$, we can express $l_{\xi_\perp}$ from the one of these expressions

$$l_{\xi_\perp} = \pm \frac{l_{v_z}}{H} = \pm \frac{l_{a_z}}{\overline{q}} \tag{118}$$

and put it into the other relation. As a result we could find the relation between the absolute values of a_z and v_z

$$l_{a_z} = \frac{\overline{q}}{H} \cdot l_{v_z} \quad . \tag{119}$$

Therefore, the absolute value l_{a_z} of the centrifugal (centripetal) acceleration a_z could be related to the absolute value l_{v_z} of the centrifugal (centripetal) velocity v_z by means of the Hubble's function H and the acceleration parameter $\overline{q}$. Then

$$a_z = \pm \frac{\overline{q}}{H} \cdot l_{v_z} \cdot n_\perp = \frac{\overline{q}}{H} \cdot v_z \quad , \tag{120}$$

$$\overline{q} = H \cdot \frac{l_{a_z}}{l_{v_z}} = \pm \frac{l_{a_z}}{l_{\xi_\perp}} \quad . \tag{121}$$

Remark. In Einstein's theory of gravitation, on the basis of cosmological models with Robertson-Walker metrics, the acceleration parameter is introduced as [5]

$$q \sim \frac{1}{H} \cdot \frac{l_{a_z}}{l_{v_z}} \ ,$$

where $l_{a_z} = \ddot{K}$, $H = \dot{K}/K$, $l_{v_z} = \dot{K}$, $l_{\xi_\perp} = K$. A comparison with the expression for $\overline{q}$ shows that $\overline{q} \sim q \cdot H^2$, and $q \sim \overline{q}/H^2$. It should be stressed that the acceleration parameter $\overline{q}$ is introduced on a pure kinematic basis and is not depending on a special type of gravitational theory in a $(\overline{L}_n, g)$-space (Einstein's theory of gravitation could be considered as a gravitational theory in a special type of a $(\overline{L}_n, g)$-space with $n = 4$ and $(\overline{L}_n, g) \equiv V_4$).

In a theory of gravitation the centripetal (relative) acceleration could be interpreted as gravitational acceleration.

Special case: ${}_sD := 0$ (shear-free relative acceleration).

$$a_z = \frac{1}{n-1} \cdot U \ \cdot \xi_\perp \quad . \tag{122}$$

If the expansion acceleration invariant $U > 0$ the acceleration a_z is a centrifugal (or expansion) acceleration. If $U < 0$ the acceleration a_z is a centripetal (or contraction) acceleration. Therefore, in the case of a shear-free relative acceleration the centrifugal or the centripetal acceleration is proportional to the expansion acceleration invariant U.

3.2 Coriolis' acceleration

The vector field a_c, defined as

$$a_c = \overline{g}[h_{\xi_\perp}({}_{rel}a)] \ ,$$

is called Coriolis' (relative) acceleration. On the basis of its definition, the Coriolis acceleration has well defined properties.

(a) The Coriolis acceleration is orthogonal to the vector field u, i.e.

$$g(u, a_c) = 0. \tag{123}$$

Proof: From

$$g(u, a_c) = g(u, \overline{g}[h_{\xi_\perp}({}_{rel}a)]) = (u)(h_{\xi_\perp})({}_{rel}a) = h_{\xi_\perp}(u, {}_{rel}a)$$

and

$$(u)(h_{\xi_\perp}) = (h_{\xi_\perp})(u) = g(u) \quad , \qquad g(u, {}_{rel}a) = 0 \quad , \tag{124}$$

it follows that

$$g(u, a_c) = [g(u)]({}_{rel}a) = g(u, {}_{rel}a) = 0.$$

(b) The Coriolis acceleration a_c is orthogonal to the centrifugal (centripetal) acceleration a_z, i.e.

$$g(a_c, a_z) = 0 \quad . \tag{125}$$

(c) The Coriolis acceleration a_c is orthogonal to the deviation vector $\xi_\perp$, i.e.

$$g(\xi_\perp, a_c) = 0 \quad . \tag{126}$$

(d) The length $\sqrt{\mid a_c^2 \mid} = \sqrt{\mid g(a_c, a_c) \mid}$ of a_c could be found by the use of the relations

$$a_c = \overline{g}[h_{\xi_\perp}({}_{rel}a)] \ ,$$

$$[g(\xi_\perp)]({}_{rel}a) = g(\xi_\perp, {}_{rel}a) \quad , \tag{127}$$

$$h_{\xi_\perp}({}_{rel}a) = g({}_{rel}a) - \frac{g(\xi_\perp, {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \quad , \tag{128}$$

$$a_c = \overline{g}[h_{\xi_\perp}({}_{rel}a)] = {}_{rel}a - a_z = \tag{129}$$

$$= {}_{rel}a - \frac{g(\xi_\perp, {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \quad . \tag{130}$$

For a_c^2 we obtain

$$\begin{aligned} a_c^2 = g(a_c, a_c) = g({}_{rel}a - a_z,\ {}_{rel}a - a_z) = \\ = g({}_{rel}a, {}_{rel}a) + g(a_z, a_z) - 2 \cdot g(a_z, {}_{rel}a) \quad . \end{aligned} \tag{131}$$

Since

$$g(a_z, a_z) = g(a_z, {}_{rel}a) = \frac{[g(\xi_\perp, {}_{rel}a)]^2}{g(\xi_\perp, \xi_\perp)} \quad , \tag{132}$$

$$a_c^2 = g({}_{rel}a, {}_{rel}a) - g(a_z, a_z) = {}_{rel}a^2 - a_z^2 \quad , \tag{133}$$

$$\begin{aligned} g({}_{rel}a, {}_{rel}a) = {}_{rel}a^2 = g(\overline{g}[A(\xi_\perp)], \overline{g}[A(\xi_\perp)]) = \\ = \overline{g}(A(\xi_\perp), A(\xi_\perp)) \ , \\ A = {}_sD + W + \frac{1}{n-1} \cdot U \cdot h_u \quad , \\ A(\xi_\perp) = {}_sD(\xi_\perp) + W(\xi_\perp) + \frac{1}{n-1} \cdot U \cdot g(\xi_\perp) \quad , \\ h_u(\xi_\perp) = g(\xi_\perp) \quad , \end{aligned}$$

$$\overline{g}({}_sD(\xi_\perp), h_u(\xi_\perp)) = {}_sD(\xi_\perp, \xi_\perp) \quad , \tag{134}$$

$$\overline{g}(W(\xi_\perp), h_u(\xi_\perp)) = W(\xi_\perp, \xi_\perp) = 0 \quad , \tag{135}$$

$$\overline{g}(h_u(\xi_\perp), h_u(\xi_\perp)) = h_u(\xi_\perp, \xi_\perp) = g\ (\xi_\perp, \xi_\perp) \ , \tag{136}$$

it follows for ${}_{rel}a^2$

$$\begin{aligned} {}_{rel}a^2 = g({}_{rel}a, {}_{rel}a) = \overline{g}({}_sD(\xi_\perp), {}_sD(\xi_\perp)) + \overline{g}(W(\xi_\perp), W(\xi_\perp)) + \\ + 2 \cdot \overline{g}({}_sD(\xi_\perp), W(\xi_\perp)) + \\ + \frac{2}{n-1} \cdot U \cdot {}_sD(\xi_\perp, \xi_\perp) + \frac{1}{(n-1)^2} \cdot U^2 \cdot g\ (\xi_\perp, \xi_\perp) \ . \end{aligned} \tag{137}$$

On the other side,

$$a_z^2 = \frac{[{}_sD(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} + \frac{1}{(n-1)^2} \cdot U^2 \cdot g(\xi_\perp, \xi_\perp) + \frac{2}{n-1} \cdot U \cdot {}_sD(\xi_\perp, \xi_\perp) \ .$$

Therefore,

$$\begin{aligned} a_c^2 = {}_{rel}a^2 - a_z^2 = \\ = \overline{g}({}_sD(\xi_\perp), {}_sD(\xi_\perp)) - \frac{[{}_sD(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} + \\ + \overline{g}(W(\xi_\perp), W(\xi_\perp)) + 2 \cdot \overline{g}({}_sD(\xi_\perp), W(\xi_\perp)) \quad . \end{aligned} \tag{138}$$

Special case: ${}_sD := 0$ (shear-free acceleration).

$$a_c^2 = \overline{g}(W(\xi_\perp), W(\xi_\perp)) \quad . \tag{139}$$

Special case: $W := 0$ (rotation-free acceleration).

$$a_c^2 = \overline{g}({}_sD(\xi_\perp),{}_s D(\xi_\perp)) - \frac{[{}_sD(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} \quad . \tag{140}$$

The explicit form of a_c could be found by the use of the relations:

$$a_c = \overline{g}[h_{\xi_\perp}({}_{rel}a)] = {}_{rel}a - \frac{g(\xi_\perp, {}_{rel}a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = {}_{rel}a - a_z =$$

$$= \overline{g}[{}_sD(\xi_\perp)] - \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \overline{g}[W(\xi_\perp)] = \tag{141}$$

$$= \overline{g}[{}_sD(\xi_\perp)] \mp {}_sD(n_\perp, n_\perp) \cdot \xi_\perp + \overline{g}[W(\xi_\perp)] \ . \tag{142}$$

Therefore, the Coriolis acceleration a_c does not depend on the expansion acceleration invariant U.

Special case: ${}_sD := 0$ (shear-free acceleration).

$$a_c = \overline{g}[W(\xi_\perp)] \quad . \tag{143}$$

Special case: $W := 0$ (rotation-free acceleration).

$$a_c = \overline{g}[{}_sD(\xi_\perp)] - \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \tag{144}$$

$$= \overline{g}[{}_sD(\xi_\perp)] \mp {}_sD(n_\perp, n_\perp) \cdot \xi_\perp \quad . \tag{145}$$

The Coriolis acceleration depends on the shear acceleration ${}_sD$ and on the rotation acceleration W. These types of accelerations generate a Coriolis acceleration between particles or mass elements in a flow.

4 Centrifugal (centripetal) acceleration as gravitational acceleration

1. The main idea of the Einstein theory of gravitation (ETG) is the identification of the centripetal acceleration with the gravitational acceleration. The weak equivalence principle stays that a gravitational acceleration could be identified with a centripetal acceleration and vice versa. From this point of view, it is worth to be investigated the relation between the centrifugal (centripetal) acceleration and the Einstein theory of gravitation as well as the possibility for describing the gravitational interaction as result of the centrifugal (centripetal) acceleration generated by the motion of mass elements (particles).

The structure of the centrifugal (centripetal) acceleration could be considered on the basis of its explicit form expressed by means of the kinematic characteristics of the relative acceleration and the relative velocity. The centrifugal (centripetal) acceleration is written as

$$a_z = [\frac{1}{n-1} \cdot U + \frac{{}_sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)}] \cdot \xi_\perp \ .$$

The vector field $\xi_{\perp}$ is directed outside of the trajectory of a mass element (particle). If the mass element generates a gravitational field the acceleration a_z should be in the direction to the mass element. If the mass element moves in an external gravitational field caused by an other gravitational source then the acceleration a_z should be directed to the source. The expansion (contraction) invariant U could be expressed by means of the kinematic characteristics of the relative acceleration or of the relative velocity in the forms [3], [9]

$$U = U_0 + I = \ _FU_0 - \ _TU_0 + I \ \ , \qquad U_0 = \ _FU_0 - \ _TU_0 \ , \tag{146}$$

where $_FU_0$ is the torsion-free and curvature-free expansion acceleration, $_TU_0$ is the expansion acceleration induced by the torsion and I is the expansion acceleration induced by the curvature, U_0 is the curvature-free expansion acceleration

$$_FU_0 = g[b] - \frac{1}{e} \cdot g(u, \nabla_u a) \ , \tag{147}$$

$$_FU_0 = a^k \ _{;k} - \frac{1}{e} \cdot g_{\overline{kl}} \cdot u^k \cdot a^l \ _{;m} \cdot u^m \ , \tag{148}$$

$$\begin{aligned} U_0 = g[b] - \overline{g}[_sP(\overline{g})\sigma] - \overline{g}[Q(\overline{g})\omega] - \dot{\theta}_1 - \frac{1}{n-1} \cdot \theta_1 \cdot \theta - \\ - \frac{1}{e} \cdot [g(u, T(a,u)) + g(u, \nabla_u a)] \ , \end{aligned} \tag{149}$$

$$I = R_{ij} \cdot u^i \cdot u^j \ \ , \qquad g[b] = g_{\overline{ij}} \cdot b^{ij} = g_{\overline{ij}} \cdot a^i \ _{;n} \cdot g^{nj} = a^n \ _{;n} \ . \tag{150}$$

2. In the ETG only the term I is used on the basis of the Einstein equations. The invariant I represents an invariant generalization of Newton's gravitational law [13]. In a V_n-space ($n = 4$) of the ETG, a free moving spinless test particle with $a = 0$ will have an expansion (contraction) acceleration $U = I$ ($U_0 = 0$) if $R_{ij} \neq 0$ and $U = I = 0$ if $R_{ij} = 0$. At the same time, in a V_n-space (the bars over the indices should be omitted)

$$_sD = \ _sM \ , \qquad _sD_0 = 0 \ , \qquad M = h_u(K_s)h_u \ , \tag{151}$$

$$\begin{aligned} M_{ij} &= h_{i\overline{k}} \cdot K_s^{kl} \cdot h_{\overline{l}j} = \frac{1}{2} \cdot h_{i\overline{k}} \cdot (K_s^{kl} + K_s^{lk}) \cdot h_{\overline{l}j} = \\ &= \frac{1}{2} \cdot h_{i\overline{k}} \cdot (R^k \ _{mnr} \cdot u^m \cdot u^n \cdot g^{rl} + R^l \ _{mnr} \cdot u^m \cdot u^n \cdot g^{rk}) \cdot h_{\overline{l}j} = \\ &= \frac{1}{2} \cdot u^m \cdot u^n \cdot (h_{i\overline{k}} \cdot R^k \ _{mnr} \cdot g^{rl} \cdot h_{\overline{l}j} + h_{i\overline{k}} \cdot R^l \ _{mnr} \cdot g^{rk} \cdot h_{\overline{l}j}) = \\ &= \frac{1}{2} \cdot u^m \cdot u^n \cdot [h_{i\overline{k}} \cdot R^k \ _{mnr} \cdot g^{rl} \cdot (g_{\overline{l}j} - \frac{1}{e} \cdot u_{\overline{l}} \cdot u_j) + \\ &+ h_{\overline{l}j} \cdot R^k \ _{mnr} \cdot g^{rk} \cdot (g_{\overline{k}i} - \frac{1}{e} \cdot u_{\overline{k}} \cdot u_i) \ \ . \end{aligned} \tag{152}$$

Since

$$h_{i\overline{k}} \cdot R^k \ _{mnr} \cdot g^{rl} \cdot (g_{\overline{l}j} - \frac{1}{e} \cdot u_{\overline{l}} \cdot u_j) =$$

$$= (g_{i\overline{k}} - \frac{1}{e} \cdot u_i \cdot u_{\overline{k}}) \cdot R^k{}_{mnr} \cdot g^{rl} \cdot g_{\overline{l}j} -$$
$$- \frac{1}{e} \cdot (g_{i\overline{k}} - \frac{1}{e} \cdot u_i \cdot u_{\overline{k}}) \cdot R^k{}_{mnr} \cdot g^{rl} \cdot u_{\overline{l}} \cdot u_j \quad ,$$

$$R^k{}_{mnr} \cdot g^{rl} \cdot u_{\overline{l}} = R^k{}_{mnr} \cdot u^r \quad ,$$
$$u_{\overline{k}} \cdot R^k{}_{mnr} = g_{\overline{k}\overline{s}} \cdot u^s \cdot R^k{}_{mnr} \quad ,$$

we have

$$\begin{aligned} h_{i\overline{k}} \cdot R^k{}_{mnr} \cdot g^{rl} \cdot h_{\overline{l}j} = {} & g_{i\overline{k}} \cdot R^k{}_{mnr} \cdot g^{rl} \cdot g_{\overline{l}j} - \\ & - \frac{1}{e} \cdot u_i \cdot g_{\overline{k}\overline{s}} \cdot u^s \cdot R^k{}_{mnr} \cdot g^{rl} \cdot g_{\overline{l}j} - \\ & - \frac{1}{e} \cdot g_{i\overline{k}} \cdot R^k{}_{mnr} \cdot g^{rl} \cdot g_{\overline{l}\overline{s}} \cdot u^s \cdot u_j + \\ & + \frac{1}{e^2} \cdot u_i \cdot g_{\overline{k}\overline{s}} \cdot u^s \cdot R^k{}_{mnr} \cdot g^{rl} \cdot g_{\overline{l}\overline{q}} \cdot u^q \cdot u_j \ . \end{aligned} \tag{153}$$

Special case: V_n-space: $S := C$.

$$M_{ij} = R_{imnj} \cdot u^m \cdot u^n \qquad . \tag{154}$$

$$\begin{aligned} {}_sD_{ij} = {}_sM_{ij} = M_{ij} - \frac{1}{n-1} \cdot I \cdot h_{ij} = \\ = R_{imnj} \cdot u^m \cdot u^n - \frac{1}{n-1} \cdot R_{mn} \cdot u^m \cdot u^n \cdot h_{ij} \ , \\ {}_sD_{ij} = (R_{imnj} - \frac{1}{n-1} \cdot R_{mn} \cdot h_{ij} \ ) \cdot u^m \cdot u^n = {}_sM_{ij} \ , \\ U = I = R_{ij} \cdot u^i \cdot u^j \ . \end{aligned} \tag{155}$$

In a V_n-space the components a_z^i of the centrifugal (centripetal) acceleration a_z have the form

$$\begin{aligned} a_z^i = {} & (\frac{1}{n-1} \cdot U + \frac{{}_sD_{jk} \cdot \xi_\perp^j \cdot \xi_\perp^k}{g_{rs} \cdot \xi_\perp^r \cdot \xi_\perp^s}) \cdot \xi_\perp^i = \\ = {} & [\frac{1}{n-1} \cdot R_{mn} \cdot u^m \cdot u^n + \\ & + \frac{1}{g_{rs} \cdot \xi_\perp^r \cdot \xi_\perp^s} \cdot (R_{jmnk} - \frac{1}{n-1} \cdot R_{mn} \cdot h_{jk}) \cdot u^m \cdot u^n \cdot \xi_\perp^j \cdot \xi_\perp^k] \cdot \xi_\perp^i \ . \end{aligned} \tag{156}$$

3. If the Einstein equations in vacuum without cosmological term ($\lambda_0 = 0$) are valid, i.e. if

$$R_{ij} = 0 \quad , \quad \lambda_0 = \text{const.} = 0 \quad , \qquad n = 4 \ , \tag{157}$$

are fulfilled then

$$a_z^i = \frac{1}{g_{rs} \cdot \xi_\perp^r \cdot \xi_\perp^s} \cdot R_{jmnk} \cdot u^m \cdot u^n \cdot \xi_\perp^j \cdot \xi_\perp^k \cdot \xi_\perp^i = \tag{158}$$

$$= R_{jmnk} \cdot u^m \cdot u^n \cdot n^j \cdot n^k \cdot \xi_\perp^i = \mathbf{g} \cdot \xi_\perp^i \ , \tag{159}$$

where

$$n^j = \frac{\xi_\perp^j}{\sqrt{\mid g_{rs} \cdot \xi_\perp^r \cdot \xi_\perp^s \mid}} \ , \tag{160}$$

$$\mathbf{g} = R_{jmnk} \cdot u^m \cdot u^n \cdot n^j \cdot n^k \quad . \tag{161}$$

If a_z is a centripetal acceleration interpreted as gravitational acceleration for a free spinless test particles moving in an external gravitational field ($R_{ij} = 0$) then the condition $\mathbf{g} < 0$ should be valid if the centripetal acceleration is directed to the particle. If the centripetal acceleration is directed to the gravitational source (in the direction $\xi_\perp$) then $\mathbf{g} > 0$.

From a more general point of view as that in the ETG, a gravitational theory could be worked out in a $(\overline{L}_n, g)$-space where a_z could also be interpreted as gravitational acceleration of mass elements or particles generating a gravitational field by themselves and caused by their motions in space-time.

4. If we consider a frame of reference in which a mass element (particle) is at rest then $u^i = g_4^i \cdot u^4$ and $\xi_\perp^i = g_a^i \cdot \xi_\perp^a := g_1^i \cdot \xi_\perp^1$. The centrifugal (centripetal) acceleration a_z^i could be written in the form

$$a_z^i = R_{1441} \cdot u^4 \cdot u^4 \cdot n^1 \cdot n^1 \cdot \xi_\perp^i = R_{1441} \cdot (u^4)^2 \cdot (n^1)^2 \cdot \xi_\perp^i \quad . \tag{162}$$

For the Schwarzschild metric

$$ds^2 = \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 \cdot (d\theta^2 + \sin^2\theta \cdot d\varphi^2) - (1 - \frac{r_g}{r}) \cdot (dx^4)^2 \quad , \tag{163}$$

$$r_g = \frac{2 \cdot k \cdot M_0}{c^2} \ , \tag{164}$$

the component $R_{1441} = g_{11} \cdot R^1{}_{441}$ of the curvature tensor has the form [5]

$$R_{1441} = -g_{11} \cdot (\Gamma^1_{44,1} - \Gamma^1_{14,4} + \Gamma^1_{11} \cdot \Gamma^1_{44} + \Gamma^1_{14} \cdot \Gamma^4_{44} - \Gamma^1_{14} \cdot \Gamma^1_{14} - \Gamma^1_{44} \cdot \Gamma^4_{41}) \ . \tag{165}$$

After introducing in the last expression the explicit form of the metric and of the Christoffel symbols Γ^i_{jk}, it follows for R_{1441}

$$R_{1441} = \frac{r_g}{r^3} \quad . \tag{166}$$

Then

$$a_z^i = \frac{r_g}{r^3} \cdot (u^4)^2 \cdot (n^1)^2 \cdot \xi_\perp^i \ . \tag{167}$$

If the co-ordinate time $t = x^4/c$ is chosen as equal to the proper time τ of the particle, i.e. if $t = \tau$ then

$$u^4 = \frac{dx^4}{d\tau} = c \cdot \frac{d\tau}{d\tau} = c \; , \qquad n^1 = 1 \quad , \qquad \xi^i_\perp = g^i_1 \cdot \xi^1_\perp \quad , \tag{168}$$

$$a^i_z = \frac{r_g}{r^3} \cdot c^2 \cdot g^i_1 \cdot \xi^1_\perp \; , \qquad \frac{r_g}{r^3} \cdot c^2 > 0 \quad . \tag{169}$$

For the centrifugal (centripetal) acceleration we obtain

$$a^1_z = \frac{2 \cdot k \cdot M_0}{r^3} \cdot \xi^1_\perp \quad , \tag{170}$$

which is exactly the relative gravitational acceleration between two mass elements (particles) with co-ordinates $x_{c1} = r$ and $x_{c2} = r + \xi^1_\perp$ [14].

Therefore, the centrifugal (centripetal) acceleration could be used for working out of a theory of gravitation in a space with affine connections and metrics as this has been done in the Einstein theory of gravitation.

5 Conclusions

In the present paper the notions of (relative) centrifugal (centripetal) and (relative) Coriolis' velocities and accelerations are introduced and considered in spaces with affine connections and metrics as velocities and accelerations of flows of mass elements (particles) moving in space-time. It is shown that these types of velocities and accelerations are generated by the relative motions between the mass elements. The centrifugal (centripetal) velocity is found to be in connection with the Hubble law in spaces with affine connections and metrics. [The relations between null vector fields and the Hubble and the Doppler effects will be considered elsewhere [17].] The accelerations are closely related to the kinematic characteristics of the relative velocity and relative acceleration. The centrifugal (centripetal) acceleration could be interpreted as gravitational acceleration as it has been done in the Einstein theory of gravitation. This fact could be used as a basis for working out of new gravitational theories in spaces with affine connections and metrics.

References

[1] S. Manoff: "Mechanics of continuous media in $(\overline{L}_n, g)$-spaces. 1. Introduction and mathematical tools" (*Preprint* gr-qc /0203016), 2002.

[2] S. Manoff: "Mechanics of continuous media in $(\overline{L}_n, g)$-spaces. 2. Relative velocity and deformations" (*Preprint* gr-qc /0203017), 2002.

[3] S. Manoff: "Mechanics of continuous media in $(\overline{L}_n, g)$-spaces. 3. Relative accelerations" (*Preprint* gr-qc /0204003), 2002.

[4] S. Manoff: "Mechanics of continuous media in $(\overline{L}_n, g)$-spaces. 4. Stress (tension) tensor". (*Preprint* gr-qc /0204004), 2002.

[5] H. Stephani: *"Allgemeine Relativitaetstheorie"*. VEB Deutscher Verlag d. Wissenschaften, Berlin, 1977, pp. 75-76.

[6] J. Ehlers: "Beitraege zur relativistischen Mechanik kontinuierlicher Medien". *Abhandlungen d. Mainzer Akademie d. Wissenschaften, Math.-Naturwiss. Kl.* Nr. **11** (1961), 792-837.

[7] Cl. Laemmerzahl: "A Characterisation of the Weylian structure of Space-Time by Means of Low Velocity Tests" (*Preprint* gr-qc /0103047), 2001.

[8] R. L. Bishop, S. I. Goldberg. *"Tensor Analysis on Manifolds"*. The Macmillan Company, New York, 1968.

[9] S. Manoff: "Kinematics of vector fields". *Complex Structures and Vector Fields* ed St Dimiev and K Sekigawa. World Scientific, Singapore, 1995, pp 61-113.

[10] S. Manoff: "Spaces with contravariant and covariant affine connections and metrics". *Phys. Part. Nuclei* **30** (1999) 5, 527-49 (Russian edn: **30** (1999) 5, 1211-69).

[11] S. Manoff: *"Geometry and Mechanics in Different Models of Space-Time: Geometry and Kinematics"*. Nova Science Publishers, New York, 2002.

[12] S. Manoff: *"Geometry and Mechanics in Different Models of Space-Time: Dynamics and Applications"*. Nova Science Publishers, New York, 2002.

[13] S. Manoff: "Einstein's theory of gravitation as a Lagrangian theory for tensor fields". *Intern. J. Mod. Phys.* **A 13** (1998) 12, 1941-67.

[14] S. Manoff: "About the motion of test particles in an external gravitational field". *Exp. Technik der Physik* **24** (1976) 5, 425-431 (in German).

[15] Ch. W. Misner, K. S. Thorne, J. A. Wheeler: *"Gravitation"*. W. H. Freeman and Company, San Francisco, 1973. (Russian translation 1977 Mir, Moscow, Vol 1, Vol. 2, Vol. 3).

[16] S. Manoff: "Flows and particles with shear-free and expansion-free velocities in (L_n, g)- and Weyl's spaces". *Class. Quantum Grav.* **19** (2002) 16, 4377-98 (*Preprint* gr-qc/0207060).

[17] S. Manoff: "Centrifugal (centripetal), Coriolis' velocities, accelerations and Hubble's law in spaces with affine connections and metrics" (*Preprint* gr-qc/0212038), 2002.